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Two order parameters in quantum XZ spin models with Gibbsian ground states

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Abstract
We describe a family of quantum spin models which are generators of a discrete Markovian process. We show that there exists an explicit expression for the ground state of such models and give a simple argument for the existence of two types of long-range order in such systems. Two special examples of these systems are analysed in detail.

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1. Introduction

The existence of long-range order for order parameters in quantum many-body systems is an important problem which is the first step towards a complete description of the phase diagram.

This problem has been solved for a large class of quantum spin systems of the mean-field type. These models include the Vonsovsky–Zener type fermion-spin systems [1] explaining the occurrence of superconductivity and of ferromagnetism at non-zero temperatures. The first rigorous analysis [1–3] of such systems made use of the so-called approximating Hamiltonian method. Other methods include large-deviation theory combined with group representations [4–7] and C^*-algebra analysis [8–10]. Note also that the approximating Hamiltonian method has been extended to boson systems in [11, 12].

Tian [21] formulated a sufficient condition for the coexistence of two independent order parameters with long-range order in the ground state of some boson and fermion systems. For the Hubbard model this condition coincides with the resonating valence bond (RVB) long-range order and on-site-pairing long-range order. Macris and Piguet [20] proved the existence of two order parameters for lattice boson–fermion systems at a non-zero temperature by generalizing [19] the Tian technique in and the Lieb–Simon reflection-positivity technique.

In this paper, we formulate a special class of quantum spin XZ models on the hypercubic lattice Z^d with a Gibbsian ground state in which long-range order occurs for the spin operators S^1 and S^3 in dimensions greater than 1. (In one-dimensional systems ferromagnetic long-range order for S^1 is easy to prove.)

Our systems differ from the XZ spin $\frac{1}{2}$ systems which admit Gibbsian ground states considered in [15]. There, the classical Gibbsian system which generates the ground state is in fact quite complicated. Kirkwood and Thomas proved that there is ferromagnetic long-range order for S^3 in the ground state in some of their ferromagnetic systems. Our proof of the S^1-long-range order is analogous to theirs. In [16] the Kirkwood–Thomas analysis is formulated as a fixed-point problem and applied to find quasi-particle states. The method has been further generalized by Yarotsky [17]. Our analysis is less general but has the advantage of simplicity.

In [18], Matsui showed that in one dimension, classical Gibbsian systems are associated with quantum Potts systems. The structure of the Matsui Hamiltonians is a special case of the Hamiltonians of XZ spin systems considered here, which can be represented as a sum of a diagonal part of a specific form and an Ising-type non-diagonal part.

Our Hamiltonians are expressed in terms of the Pauli matrices

$$S^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \text{and} \qquad S^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

Given a finite subset $\Lambda \subset \mathrm{Z}^d$ with cardinality $|\Lambda|$ let S_x^1 etc be the corresponding operators on $\mathbb{E}_\Lambda = (\mathbb{C}^2)^\Lambda$ acting on the factor for the point $x \in \Lambda$. If we denote for $s_\Lambda \in \{-1, 1\}^\Lambda$,

$$\Psi_\Lambda^0(s_\Lambda) = \otimes_{x\in\Lambda}\psi_0(s_x) \qquad \text{where} \quad \psi_0(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \psi_0(-1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then this can be written as

$$S_x^1\Psi_\Lambda^0(s_\Lambda) = \Psi_\Lambda^0\left(s_\Lambda^{\{x\}}\right) \qquad S_x^3\Psi_\Lambda^0(s_\Lambda) = s_x\Psi_\Lambda^0(s_\Lambda) \tag{2}$$

where, for any subset $A \subset \Lambda$, s_Λ^A is the configuration s_Λ with the spins in A flipped. (Note that the states $\Psi_\Lambda^0(s_\Lambda)$ form an orthonormal basis for $(\mathbb{C}^2)^\Lambda$. In particular,

$$\left\langle\Psi_\Lambda^0(s_\Lambda)\middle|\Psi_\Lambda^0(s_\Lambda)\right\rangle = \delta(s_\Lambda; s'_\Lambda) = \prod_{x\in\Lambda}\delta_{s_x,s'_x}$$

where δ_{s_x,s'_x} is the Kronecker symbol.)

We now define the operators

$$P_A = S_A^1 - \mathrm{e}^{-\frac{\alpha}{2}W_A(S_\Lambda^3)} \qquad S_A^1 = \prod_{x\in A}S_x^1 \tag{3}$$

where

$$W_A(s_\Lambda) = U_0\left(s_\Lambda^A\right) - U_0(s_\Lambda) \qquad U_0\left(s_\Lambda^A\right) = U_0(s_{\Lambda\setminus A}, -s_A). \tag{4}$$

We tacitly assume that the function U_0 satisfy all the conditions needed for the existence of the thermodynamic limit.

Our main results concern Hamiltonians of the form

$$H_\Lambda = \sum_{A\subset\Lambda}J_A P_A \qquad J_A \leqslant 0 \tag{5}$$

In theorem 2.1, we show that their ground state is given by

$$\Psi_\Lambda = \sum_{s_\Lambda}\mathrm{e}^{-\frac{\alpha}{2}U_0(s_\Lambda)}\Psi_\Lambda^0(s_\Lambda) \qquad \alpha \in \mathrm{R}^+. \tag{6}$$

In the proof we establish that the Hamiltonian (5) is the generator of a discrete Markovian process. The spectral structure for such generators in the simplest case ($|A| = 1$) was established in [22]. In theorem 2.2, we formulate conditions on J_A for which this ground state is unique. As a simple consequence, we show in theorem 2.3 that in dimensions $d > 1$, there are two types of long-range order in these systems.

In the third section, we calculate explicit expressions for the Hamiltonians in the case $J_A = 0, |A| > 2$ and with the simplest choice of a ferromagnetic U_0. The Hamiltonian corresponding to the case $d = 1, J_A = 0, |A| > 1$ already appeared in [18]. The case $J_A = 0, |A| \neq 2$ is interesting since our Hamiltonian is expressed as a perturbation of the simple ferromagnetic Hamiltonian

$$H_\Lambda = J \sum_{\langle x,y\rangle \in \Lambda} \left(S_x^1 S_y^1 + \gamma S_x^3 S_y^3\right) \qquad J < 0$$

where $\gamma = 4d(\cosh\alpha)^{4d-3}\sinh\alpha$. Our condition of uniqueness of the ground state does not apply to this case since it does not hold if $J_A = 0$ for all A with $|A| \neq 2$. However, see remark 2.2.

Remark. The class of Hamiltonians for which (6) is a ground state can be generalized to

$$H_\Lambda = \sum_{A_1,\ldots,A_l \subset \Lambda} J_{A_{(l)}} \left(P_{A_1} \ldots P_{A_l} + P_{A_l} \ldots P_{A_1}\right) \qquad A_{(l)} = (A_1, \ldots, A_l) \tag{7}$$

where the summation is over families of disjoint non-empty subsets of Λ. This follows from the following equality for an arbitrary A:

$$P_A \Psi_\Lambda = 0. \tag{8}$$

2. Main results

We first prove that (6) is a ground state with eigenvalue zero for the Hamiltonian (5).

Theorem 2.1. *The Hamiltonian (5) is a positive self-adjoint operator on* $(\mathbb{C}^2)^\Lambda$ *and the state* Ψ_Λ*, given by (6), is a ground state with eigenvalue zero.*

We begin by proving (8). This shows that Ψ_Λ is an eigenfunction of the Hamiltonian (5) with eigenvalue zero. The identity (8) follows easily by changing signs of the spin variables s_A in the first term:

$$\begin{aligned} P_A \Psi_\Lambda &= \sum_{s_\Lambda} \left(\Psi_\Lambda^0\left(s_\Lambda^A\right) - \mathrm{e}^{-\frac{\alpha}{2}W_A(s_\Lambda)}\Psi_\Lambda^0(s_\Lambda)\right)\mathrm{e}^{-\frac{\alpha}{2}U_0(s_\Lambda)} \\ &= \sum_{s_\Lambda} \left(\Psi_\Lambda^0\left(s_\Lambda^A\right)\mathrm{e}^{-\frac{\alpha}{2}U_0(s_\Lambda)} - \Psi_\Lambda^0(s_\Lambda)\,\mathrm{e}^{-\frac{\alpha}{2}U_0(s_\Lambda^A)}\right) \\ &= \sum_{s_\Lambda} \left(\mathrm{e}^{-\frac{\alpha}{2}U_0(s_\Lambda^A)} - \mathrm{e}^{-\frac{\alpha}{2}U_0(s_\Lambda^A)}\right)\Psi_\Lambda^0(s_\Lambda) = 0. \end{aligned}$$

Next we prove that the Hamiltonian is a positive operator. For this purpose, we define two further operators

$$H_\Lambda^+ = \mathrm{e}^{\frac{\alpha}{2}U_0(S_\Lambda^3)} H_\Lambda\, \mathrm{e}^{-\frac{\alpha}{2}U_0(S_\Lambda^3)} \qquad H_\Lambda^- = \mathrm{e}^{-\frac{\alpha}{2}U_0(S_\Lambda^3)} H_\Lambda \mathrm{e}^{\frac{\alpha}{2}U_0(S_\Lambda^3)}. \tag{9}$$

It is clear that

$$\left(H_\Lambda^+\right)^* = H_\Lambda^- \qquad H_\Lambda^- = \mathrm{e}^{-\alpha U_0(S_\Lambda^3)} H_\Lambda^+\, \mathrm{e}^{\alpha U_0(S_\Lambda^3)}. \tag{10}$$

where the star denotes the adjoint in the Hilbert space $\mathrm{E}_\Lambda = (\mathbb{C}^2)^\Lambda$.

A straightforward calculation on the basis Ψ^0_Λ shows that

$$H^+_\Lambda = \sum_{A\subseteq\Lambda} J_A\,\mathrm{e}^{-\frac{\alpha}{2}W_A(S^3_\Lambda)}\big(S^1_A - I\big) \tag{11}$$

where I is the unit operator. This operator is symmetric with respect to the new scalar product

$$\langle F'|F\rangle_{U_0} = \langle F'|\,\mathrm{e}^{-\alpha U_0(S^3_\Lambda)}F\rangle. \tag{12}$$

Indeed,

$$\begin{aligned}\langle F'|H^+_\Lambda F\rangle_{U_0} &= \langle F'|\,\mathrm{e}^{-\alpha U_0(S^3_\Lambda)}H^+_\Lambda F\rangle\\ &= \sum_{A\subseteq\Lambda} J_A\langle F'|\mathrm{e}^{-\frac{\alpha}{2}[U_0(S^3_\Lambda)+U_0(S^{3A}_\Lambda)]}\big(S^1_A - I\big)F\rangle\\ &= \sum_{A\subseteq\Lambda} J_A\langle\big(S^1_A - I\big)F'|\mathrm{e}^{-\frac{\alpha}{2}[U_0(S^3_\Lambda)+U_0(S^{3A}_\Lambda)]}F\rangle\\ &= \langle H^+_\Lambda F'|F\rangle_{U_0}.\end{aligned}$$

Here we used the equalities

$$\mathrm{e}^{-\frac{\alpha}{2}U_0(S^3_\Lambda)}S^1_A = S^1_A\,\mathrm{e}^{-\frac{\alpha}{2}U_0(S^{3A}_\Lambda)} \qquad \mathrm{e}^{-\frac{\alpha}{2}U_0(S^{3A}_\Lambda)}S^1_A = S^1_A\,\mathrm{e}^{-\frac{\alpha}{2}U_0(S^3_\Lambda)} \tag{13}$$

From these inequalities we derive, also,

$$\langle F'|H^+_\Lambda F\rangle_{U_0} = \langle\mathrm{e}^{-\frac{\alpha}{2}U_0(S^3_\Lambda)}F'|H_\Lambda\mathrm{e}^{-\frac{\alpha}{2}U_0(S^3_\Lambda)}F'\rangle. \tag{14}$$

This shows that it suffices to prove that H^+_Λ is a positive operator for the new scalar product (12). Let

$$F = \sum_{s_\Lambda} F(s_\Lambda)\Psi^0_\Lambda(s_\Lambda)$$

then

$$\big(H^+_\Lambda F\big)(s_\Lambda) = -\sum_{A\subseteq\Lambda} J_A\,\mathrm{e}^{-\frac{\alpha}{2}W_A(s_\Lambda)}\big(F(s_\Lambda) - F\big(s^A_\Lambda\big)\big). \tag{15}$$

In deriving this equality one has to once again change the signs of the spins s_A in the expansion of $H^+_\Lambda F$ on the basis Ψ^0_Λ.

This means that

$$\begin{aligned}\langle F|H^+_\Lambda F\rangle_{U_0} &= -\sum_{A\subseteq\Lambda} J_A\sum_{s_\Lambda}\mathrm{e}^{-\frac{\alpha}{2}[U_0(s_\Lambda)+U_0(s^A_\Lambda)]}\big(F(s_\Lambda) - F\big(s^A_\Lambda\big)\big)F(s_\Lambda)\\ &= -\frac{1}{2}\sum_{A\subseteq\Lambda} J_A\sum_{s_\Lambda}\mathrm{e}^{-\frac{\alpha}{2}[U_0(s_\Lambda)+U_0(s^A_\Lambda)]}\big(F(s_\Lambda) - F\big(s^A_\Lambda\big)\big)^2 \geqslant 0.\end{aligned} \tag{16}$$

Here we used the fact that the exponential weight in the sum is invariant under changing signs of spin variables s_A. It now follows that H_Λ is positive definite.

Remark 2.1. The operator H^+_Λ is an analogue of the operator generated by the Dirichlet form for continuous spins. Its exponent $\mathrm{e}^{-tH^+_\Lambda}$ generates a discrete Markov process which can be called a generalized spin-flip process. For its adjoint the following relations are valid:

$$(H^-_\Lambda F)(s_\Lambda) = \sum_{A\subseteq\Lambda} J_A\big[\mathrm{e}^{\frac{\alpha}{2}W_A(s_\Lambda)}F\big(s^A_\Lambda\big) - \mathrm{e}^{-\frac{\alpha}{2}W_A(s_\Lambda)}F(s_\Lambda)\big] \qquad \sum_{s_\Lambda}(H^-_\Lambda F)(s_\Lambda) = 0.$$

The last equality implies the validity of the law of conservation of probability and is derived after changing signs of spins s_A in the first term of the first equality $\big(W_A\big(s^A_\Lambda\big) = -W_A(s_\Lambda)\big)$.

One expects that this process is characterized by a non-equilibrium phase transition as in the case of the non-equilibrium system of interacting Brownian oscillators [23].

Uniqueness of the ground state will be derived from the Perron–Frobenius theorem [13, 14].

Theorem. *Let the square matrix B be non-negative and irreducible. Then the spectral radius $\rho(B)$ is a simple eigenvalue of B and $\rho(B) > 0$. Moreover, the components of the associated eigenvector are all strictly positive.*

We recall that a matrix is non-negative if all its matrix elements are non-negative, and an $n \times n$-matrix B is irreducible if there does not exist a subset $I \subset \{1, \ldots, n\}$ such that for all $(i, j) \in I \times I^c$, the matrix elements $B_{i,j} = 0$.

We use this theorem to derive two alternative conditions for uniqueness of the ground state.

Theorem 2.2. *The ground state Ψ_Λ of H_Λ is unique if one of the following conditions is satisfied:*

1. *$J_{\{x\}} < 0$ for all $x \in \Lambda$; or*
2. *For every pair of points $x, y \in \Lambda$ there exists a chain $x_0 = x, x_1, \ldots, x_n = y$ of points in Λ such that $J_{\{x_i, x_{i+1}\}} < 0$ and there is a set $A \subset \Lambda$ with $J_A < 0$ and $|A|$ odd.*

Proof. We apply the Perron–Frobenius theorem to the operator $-H_\Lambda + aI$, where I is the identity operator (matrix) and a is a constant given by

$$a = \sum_{A \subset \Lambda} J_A \, \mathrm{e}^{-\frac{\alpha}{2} W_A(s_\Lambda)}. \tag{17}$$

Consider first the case $J_{\{x\}} < 0$ for all $x \in \Lambda$. Suppose that $I \subset \{-1, 1\}^\Lambda$ is such that

$$\left\langle \Psi^0_\Lambda(s'_\Lambda) \middle| (-H_\Lambda + aI) \Psi^0_\Lambda(s_\Lambda) \right\rangle = -\sum_{A \subset \Lambda} J_A \left\langle \Psi^0_\Lambda(s'_\Lambda) \middle| S^1_A \Psi^0_\Lambda(s_\Lambda) \right\rangle = 0$$
$$\forall s_\Lambda \in I, s'_\Lambda \in I^c. \tag{18}$$

Since $I \neq \{-1, 1\}^\Lambda$, there exists $s_\Lambda \in I$ and $x \in \Lambda$ such that $s'_\Lambda := S^1_x \Psi^0_\Lambda(s_\Lambda) = \Psi^0_\Lambda\left(s^{\{x\}}_\Lambda\right) \notin I$. This contradicts (18) since all $J_A \leqslant 0$ and $J_{\{x\}} < 0$.

Next consider case 2, and assume again that (18) holds. Similar to the previous case, if $s_\Lambda \in I$ and $x, y \in \Lambda$ such that $J_{\{x,y\}} < 0$ then $s^{\{x,y\}}_\Lambda \in I$. By flipping pairs of spins in a chain as in the hypothesis, it then follows that we can flip any pair of spins in s_Λ. We conclude that I must contain all configurations with an even number of spins $s_x = -1$ or all configurations with an odd number of minus-spins. However, it is also assumed that there is a set $A \subset \Lambda$ with $|A|$ odd and $J_A < 0$. Flipping the spins in A converts a configuration with an odd number of spins $s_x = -1$ to one with an even number and vice versa. It follows that I must contain all configurations. □

Remark 2.2. The second condition in case 2 is not superfluous: it follows from the proof that even if $J_A < 0$ for all A with $|A| = 2$, there does exist a nontrivial set I satisfying (18). Indeed, in this case the spaces spanned by $\Psi^0_\Lambda(s_\Lambda)$ where $\#\{x : s_x = -1\}$ is odd resp. even are invariant, and the ground state is two-fold degenerate.

One of the most interesting features of the models considered is that they have two order parameters with long-range order. This is now surprisingly easy to prove.

Define, for finite subsets $A \subset \mathbb{Z}^d$, and operators F_A depending on S_x^1, S_x^2 and S_x^3 with $x \in A$,

$$\langle F_A \rangle = \lim_{\Lambda \to \mathbb{Z}^d} \langle F_A \rangle_\Lambda \qquad \langle F_A \rangle_\Lambda = \frac{(\Psi_\Lambda | F_A \Psi_\Lambda)}{\langle \Psi_\Lambda, \Psi_\Lambda \rangle} \tag{19}$$

where Ψ_Λ is the ground state. The Gibbsian nature of the ground state then immediately yields the following theorem.

Theorem 2.3. *Suppose that the Hamiltonian H_Λ of a quantum spin system on finite subsets of the lattice $\mathbb{Z}^d$ is given by (5) and that $\lim_{\Lambda \to \mathbb{Z}^d} W_A(s_\Lambda)$ exists for all finite $A \subset \mathbb{Z}^d$. Suppose moreover that the limit is bounded if $|A| = 2$. Then, for $d \geqslant 1$, there is ferromagnetic long-range order for S^1. Moreover, if there is long-range order in the corresponding classical spin system with the potential energy U_0 then such long-range order occurs also for S^3 in the ground state of the quantum system.*

Proof. We have to prove that

$$\left\langle S_x^1 S_y^1 \right\rangle > a \qquad \text{for} \quad a > 0. \tag{20}$$

Writing

$$Z_\Lambda = \langle \Psi_\Lambda | \Psi_\Lambda \rangle = \sum_{s_\Lambda} \mathrm{e}^{-\frac{\alpha}{2} U_0(s_\Lambda)}$$

we have

$$\left\langle S_x^1 S_y^1 \right\rangle_\Lambda = Z_\Lambda^{-1} \sum_{s_\Lambda} \mathrm{e}^{-\frac{\alpha}{2} U_0(s_\Lambda)} \, \mathrm{e}^{-\frac{\alpha}{2} W_{x,y}(s_\Lambda)} \geqslant \inf_{s_\Lambda, x, y} \mathrm{e}^{-\frac{\alpha}{2} W_{x,y}(s_\Lambda)} < +\infty.$$

This proves (20).

Since S^3 is a diagonal matrix, the ground-state expectation value of a function of S_x^3 equals the classical Gibbsian expectation value of the function depending on classical spins. This proves the last statement of the theorem. □

Remark 2.3. For short-range interactions the condition for $W_{x,y}$ of the theorem is always satisfied. It is well known that for a ferromagnetic nearest-neighbour pair interaction

$$U_0(s_\Lambda) = -g \sum_{\langle x,y \rangle \subseteq \Lambda} s_x s_y \qquad (g > 0) \tag{21}$$

there is ferromagnetic long-range order in the classical system at sufficiently low temperatures.

3. Examples

In this section, we show that some of the Hamiltonians considered in the previous section have the following form:

$$H_\Lambda = \tilde{H}_\Lambda + H_{\partial\Lambda} + |\Lambda| \alpha_0 \tag{22}$$

where $\tilde{H}_\Lambda$ is a polynomial in S_x^1 and S_x^3, $H_{\partial\Lambda}$ is a boundary term and α_0 is a constant.

We consider two specific examples.

3.1. Example 1

Put $J_x = -1$; $J_{x_1,\ldots,x_k} = 0, k > 1$ and

$$U_0(s_\Lambda) = - \sum_{\langle x,y \rangle \in \Lambda} s_x s_y. \tag{23}$$

Then

$$W_x(s_\Lambda) = 2s_x \sum_{y\in\Lambda,|y-x|=1} s_y. \tag{24}$$

Let n_x be the number of nearest neighbours of x. Then from the simple equality

$$\mathrm{e}^{-\alpha S} = \cosh\alpha - S\sinh\alpha \qquad S^2 = I \tag{25}$$

it follows that $(Y_k = (y_1, \ldots, y_k))$

$$\begin{aligned}
\mathrm{e}^{-\frac{\alpha}{2}W_x(S^3_\Lambda)} &= \prod_{y\in\Lambda,|y-x|=1} \mathrm{e}^{-\alpha S^3_x S^3_y} \\
&= \prod_{y\in\Lambda,|y-x|=1} \big(\cosh\alpha - S^3_x S^3_y \sinh\alpha\big) \\
&= \sum_{k=1}^{[\frac{n_x}{2}]} (\sinh\alpha)^{2k} (\cosh\alpha)^{n_x-2k} \sum_{Y_{2k}\subset\Lambda,|y_j-x|=1} S^3_{[Y_{2k}]} \\
&\quad - S^3_x \sum_{k=0}^{[\frac{n_x-1}{2}]} (\sinh\alpha)^{2k+1} (\cosh\alpha)^{n_x-2k-1} \sum_{Y_{2k+1}\subset\Lambda,|y_j-x|=1} S^3_{[Y_{2k+1}]} + (\cosh\alpha)^{n_x}
\end{aligned}$$

where $[n]$ is the integer part of the number n. The Hamiltonian can therefore be written as

$$\begin{aligned}
H_\Lambda = -\sum_{x\in\Lambda} \Bigg\{ S^1_x &- \sum_{k=1}^{[\frac{n_x}{2}]} \alpha_k(n_x) \sum_{Y_{2k}\subset\Lambda,|y_j-x|=1} S^3_{[Y_{2k}]} \\
&+ \sum_{k=0}^{[\frac{n_x-1}{2}]} \beta_k(n_x) \sum_{Y_{2k+1}\subset\Lambda,|y_j-x|=1} S^3_x S^3_{[Y_{2k-1}]} \Bigg\} + (\cosh\alpha)^{2d}|\Lambda| - c_{\partial\Lambda}
\end{aligned}$$

where

$$\alpha_k(n) = (\sinh\alpha)^{2k}(\cosh\alpha)^{n-2k}$$

and

$$\beta_k(n) = (\sinh\alpha)^{2k+1}(\cosh\alpha)^{n-2k-1}$$

and

$$c_{\partial\Lambda} = (\cosh\alpha)^d(\cosh^d\alpha - 1)|\partial\Lambda|$$

is a boundary term.

It is now evident that (22) holds with $\alpha_0 = (\cosh\alpha)^{2d}$ and

$$\begin{aligned}
\tilde{H}_\Lambda = -\sum_{x\in\Lambda} S^1_x &- 2d\beta_0(2d) \sum_{\langle x,y\rangle\in\Lambda} S^3_x S^3_y + \alpha_1(2d) \sum_{x\in\Lambda} \sum_{Y_2\subset\Lambda,|y_j-x|=1} S^3_{y_1} S^3_{y_2} \\
&+ \sum_{k=2}^{d} \Bigg[\alpha_k(2d) \sum_{x\in\Lambda} \sum_{Y_{2k}\subset\Lambda,|y_j-x|=1} S^3_{[Y_{2k}]} \\
&- \beta_{k-1}(2d) \sum_{x\in\Lambda} \sum_{Y_{2k-1}\subset\Lambda,|y_j-x|=1} S^3_x S^3_{[Y_{2k-1}]} \Bigg].
\end{aligned} \tag{26}$$

In the case $d = 1$ one has in particular, for $\Lambda = [-L, L]$,

$$\tilde{H}_\Lambda = -\sum_{x\in\Lambda} S^1_x - (\sinh 2\alpha) \sum_{\langle x,y\rangle\in\Lambda} S^3_x S^3_y + (\sinh\alpha)^2 \sum_{x,y\in\Lambda,|x-y|=2} S^3_x S^3_y \tag{27}$$

with a boundary term

$$H_{\partial\Lambda} = \sinh\alpha(1-\cosh\alpha)\big(S^3_{-L}S^3_{-L+1} + S^3_{L-1}S^3_L\big) + 2\cosh\alpha(1-\cosh\alpha).$$

$\tilde{H}_\Lambda$ is essentially the Hamiltonian introduced by Matsui in [18]. Note that U_0 is of the form (21) so that in dimensions $d \geqslant 2$ there is long-range order of two different kinds by theorem 2.3.

3.2. Example 2

Put $J_x = 0$, $J_{x,y} = -1$, $|x-y| = 1$; $J_{x,y} = 0$, $|x-y| > 1$ and let U_0 be given by (23).

We first consider the one-dimensional case $d = 1$.

Since $J_A = 0$ unless A is a pair of nearest neighbour sites, we only need to compute $W_{\{x,x+1\}}$. It is given by the formula ($\Lambda = [-L, L]$)

$$W_{x,x+1}(s_\Lambda) = 2((1-\delta_{-L,x})s_{x-1}s_x + (1-\delta_{L,x})s_{x+1}s_{x+2}). \tag{28}$$

If $-L+1 \leqslant x \leqslant L-2$ then an application of (25) yields

$$\begin{aligned} \mathrm{e}^{-\frac{\alpha}{2}W_{x,x+1}(S^3_\Lambda)} &= \big(\cosh\alpha - S^3_{x-1}S^3_x\sinh\alpha\big)\big(\cosh\alpha - S^3_{x+1}S^3_{x+2}\sinh\alpha\big) \\ &= -(\cosh\alpha)(\sinh\alpha)\big(S^3_{x-1}S^3_x + S^3_{x+1}S^3_{x+2}\big) \\ &\quad + (\sinh\alpha)^2 S^3_{x-1}S^3_xS^3_{x+1}S^3_{x+2} + (\cosh\alpha)^2. \end{aligned}$$

We also have

$$\mathrm{e}^{-\frac{\alpha}{2}W_{-L,-L+1}(S^3_\Lambda)} = \cosh\alpha - S^3_{-L+1}S^3_{-L+2}\sinh\alpha$$

and

$$\mathrm{e}^{-\frac{\alpha}{2}W_{L-1,L}(S^3_\Lambda)} = \cosh\alpha - S^3_{L-2}S^3_{L-1}\sinh\alpha.$$

We thus obtain the following expression for the Hamiltonian:

$$\begin{aligned} H_\Lambda = &- \sum_{-L\leqslant x\leqslant L-1} S^1_xS^1_{x+1} - (\cosh\alpha)(\sinh\alpha)\sum_{-L+1\leqslant x\leqslant L-2}\big(S^3_{x-1}S^3_x + S^3_{x+1}S^3_{x+2}\big) \\ &+ (\sinh\alpha)^2 \sum_{-L+1\leqslant x\leqslant L-2} S^3_{[(x-1,\ldots,x+2)]} - \sinh\alpha\big(S^3_{-L+1}S^3_{-L+2} + S^3_{L-2}S^3_{L-1}\big) \\ &+ (2L-2)(\cosh\alpha)^2 + 2\cosh\alpha. \end{aligned} \tag{29}$$

This is obviously of the form (22) with $\alpha_0 = (\cosh\alpha)^2$, and bulk Hamiltonian given by

$$\tilde{H}_\Lambda = -\sum_{-L\leqslant x\leqslant L-1}\big[S^1_xS^1_{x+1} + (\sinh 2\alpha)S^3_xS^3_{x+1}\big] + (\sinh\alpha)^2\sum_{-L+1\leqslant x\leqslant L-2} S^3_{[(x-1,\ldots,x+2)]}. \tag{30}$$

Next we analyse the case of arbitrary d. We have, for a bond $\langle x, y\rangle \in \Lambda$,

$$W_{x,y}(s_\Lambda) = 2\sum_{b\in B^o_{x,y}} s_b \qquad s_b = s_zs_{z'} \qquad \text{if } \langle z, z'\rangle = b \tag{31}$$

and hence

$$\mathrm{e}^{-\frac{\alpha}{2}W_{x,y}(S^3_\Lambda)} = \prod_{\langle z,z'\rangle\in B^o_{x,y}} \mathrm{e}^{-\alpha S^3_zS^3_{z'}}. \tag{32}$$

where $B^o_{x,y}$ is the set of bonds stemming from the points x, y excluding the bond $\langle x, y\rangle$ itself. Another application of (25) yields

$$H_\Lambda = -\sum_{\langle x,y\rangle\in\Lambda} S^1_xS^1_y + \sum_{\langle x,y\rangle\in\Lambda}\left\{\left(\sum_{Z\subset N_x\setminus\{y\}}\gamma_x(|Z|)S^3_{[Z]_x}\right)\left(\sum_{Z'\subset N_y\setminus\{x\}}\gamma_y(|Z'|)S^3_{[Z']_y}\right)\right\} \tag{33}$$

where $N_x = \{z \in \Lambda||x - z| = 1\}$ and $N_y\{z \in \Lambda||y - z| = 1\}$, $[Z]_x = Z$ if $|Z|$ is even and $[Z]_x = Z \cup \{x\}$ if $|Z|$ is odd, and similarly for $[Z']_y$ and

$$\gamma_x(n) = (\cosh\alpha)^{n_x-n-1}(\sinh\alpha)^n \tag{34}$$

and similarly for γ_y. This is clearly of the form (22) with $\alpha_0 = d(\cosh\alpha)^{2(2d-1)}$, and bulk Hamiltonian given by

$$\tilde{H}_\Lambda = -\sum_{\langle x,y\rangle\in\Lambda}\left[S_x^1S_y^1 + \gamma S_x^3S_y^3\right] + \sum_{\langle x,y\rangle\in\Lambda}\sum_{j=2}^{2(2d-1)}(-1)^j\gamma_j \sum_{\{b_1,\ldots,b_j\}\subset B_{x,y}^o} S_{[\cup b_j]}^3 \tag{35}$$

where

$$\gamma = 2(2d-1)(\cosh\alpha)^{4d-3}(\sinh\alpha) \tag{36}$$

and

$$\gamma_j = (\cosh\alpha)^{4d-2-j}(\sinh\alpha)^j \tag{37}$$

and $\cup b_j$ includes x or y if they occur an odd number of times.

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